A Many–to–Many 'Rural Hospital Theorem'

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Motivation

- Matching markets: university admissions, entry-level labor markets, kidney exchange, school choice, . . .

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  - **C1**. they had difficulties to fill their posts; and
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- **Question**: can the distribution of medical graduates over residency programs be altered by using a *different* stable mechanism?

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  - **C2**. the posts they did fill were mainly staffed by graduates of foreign medical schools.
- **Question**: can the distribution of medical graduates over residency programs be altered by using a *different* stable mechanism?
- **Answer**: no. The maldistribution is not an artifact of the employed stable mechanism (Roth, 1986).

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- **R1**. The **number** of medical graduates assigned to a hospital is the **same** across **stable** matchings.

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- **R1**. The **number** of medical graduates assigned to a hospital is the **same** across **stable** matchings.
- **R2**. The **set** of medical graduates assigned to a **rural** hospital is the **same** across **stable** matchings.

This Paper

- studies many–to–many matching

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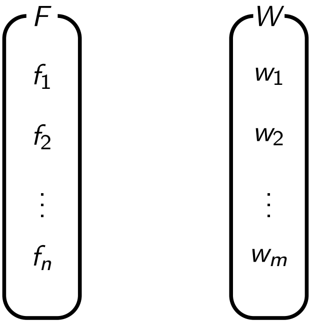
- assumes preferences are 'substitutable'
- explores pervasiveness of the rural hospital theorem
  - **R1**: **maximality** of an existing preference domain
  - **R2**: **maximality** of a **new** preference domain

Outline

- Model
- Preference Domains
- Results

Model



Model

$$
\begin{array}{c}
\mathbf{P_{f_1}} \\ \hline
w_1\, w_2\, w_3 \\
w_2\, w_3 \\
w_4 \\
\emptyset \\
\vdots
\end{array}
\qquad
\overset{F}{\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}}
\qquad\qquad
\overset{W}{\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix}}
\qquad
\begin{array}{c}
\mathbf{P_{w_1}} \\ \hline
f_1 \\
f_2\, f_3 \\
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\end{array}
$$

$P = (P_a)_{a \in F \cup W}$: preference profile

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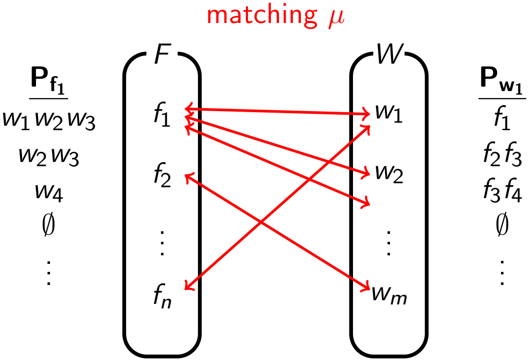

$P = (P_a)_{a \in F \cup W}$: preference profile

Model

quota ≥ 1
maximum
size of
acceptable sets
(preferred to $\emptyset$)

$\mathbf{P_{f_1}}$	F	W	$\mathbf{P_{w_1}}$
$w_1 w_2 w_3$	f_1	w_1	f_1
$w_2 w_3$	f_2	w_2	$f_2 f_3$
w_4	$\vdots$	$\vdots$	$f_3 f_4$
$\emptyset$	f_n	w_m	$\emptyset$
$\vdots$			$\vdots$

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Model

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$q_{f_1} = 3$

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$w_1 w_2 w_3$	f_1	w_1	f_1
$w_2 w_3$	f_2	w_2	$f_2 f_3$
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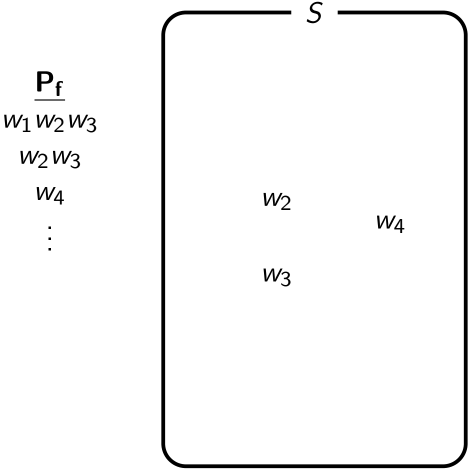
$w_2 w_3$	f_2	w_2	$f_2 f_3$
w_4	$\vdots$	$\vdots$	$f_3 f_4$
$\emptyset$	f_n	w_m	$\emptyset$
$\vdots$			$\vdots$

$q_{w_1} = 2$

$P = (P_a)_{a \in F \cup W}$: preference profile

Model

Let $S \subseteq W$.
Choice set $\mathbf{Ch(S, P_f)}$:



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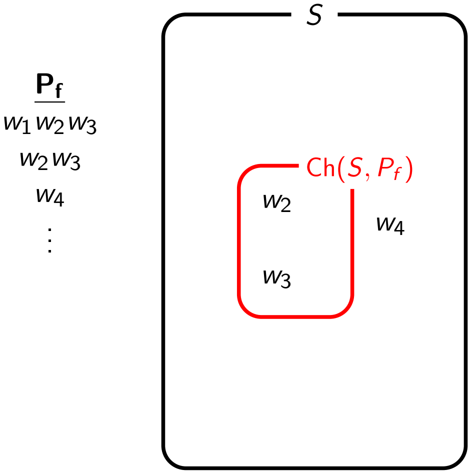
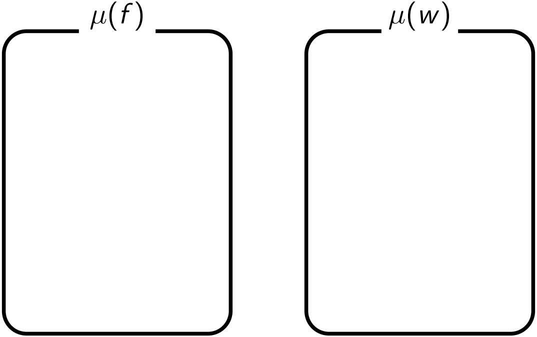


Solution Concept

Matching μ is **individually rational at P** if for each $a \in F \cup W$, $\mathrm{Ch}(\mu(a), P_a) = \mu(a)$.



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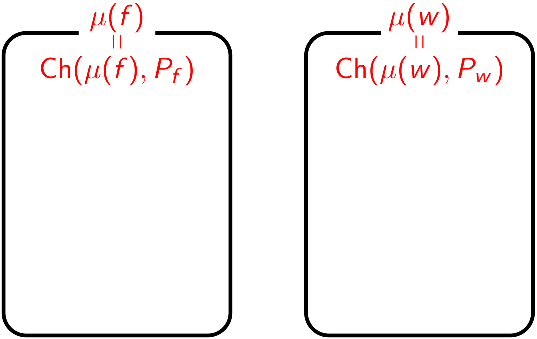
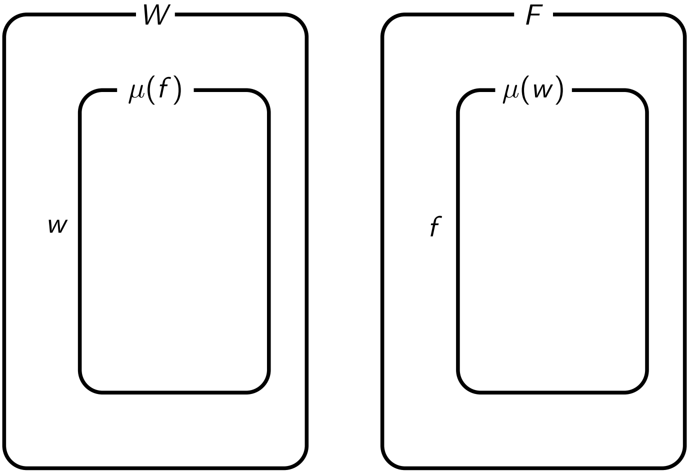


Solution Concept

Matching μ is **stable at P** if it is **individually rational at P** and **NOT blocked by a pair $(\mathbf{f}, \mathbf{w})$** in the following way:



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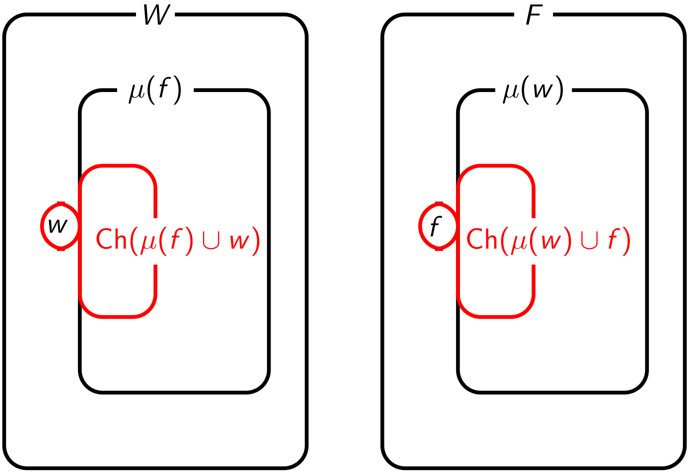

Rural Hospital Theorem

Let $\Sigma(P)$ denote the set of stable matchings at P.

- **R1**. For each pair $\mu, \mu' \in \Sigma(P)$ and each $a \in F \cup W$,
$$|\mu(a)| = |\mu'(a)|;$$

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- **R1**. For each pair $\mu, \mu' \in \Sigma(P)$ and each $a \in F \cup W$,
$$|\mu(a)| = |\mu'(a)|;$$
- **R2**. For each pair $\mu, \mu' \in \Sigma(P)$ and each $a \in F \cup W$,
$$|\mu(a)| < q_a \implies \mu(a) = \mu'(a).$$

Preference Domains

- Responsiveness
- **Substitutability**

- Cardinal Monotonicity

- Separability
- Weak Separability

- Quota Filling

Responsiveness (Roth, 1985)

For each $S \subseteq W$ with $|S| < q_f$ and each pair $w, w' \in W \setminus S$,

$S \cup w$ $\quad\quad$ $S \cup w'$

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For each $S \subseteq W$ with $|S| < q_f$ and each pair $w, w' \in W \backslash S$,

$$S \cup w \quad P_f \quad S \cup w' \quad \Longleftrightarrow \quad w \; P_f \; w'$$

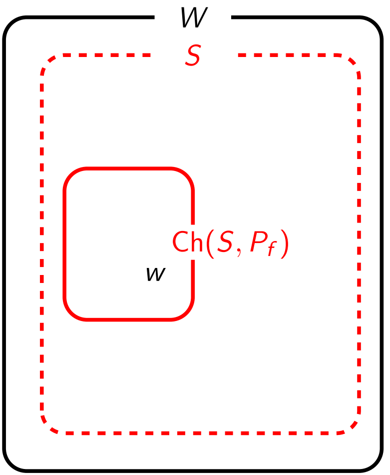
Responsiveness (Roth, 1985)

For each $S \subseteq W$ with $|S| < q_f$ and each pair $w, w' \in (W \setminus S) \cup \{\emptyset\}$,

$$S \cup w \quad P_f \quad S \cup w' \quad \Longleftrightarrow \quad w \; P_f \; w'$$

Substitutability (Kelso and Crawford, 1982)

$w \in S' \subseteq S \subseteq W$ and $w \in \mathrm{Ch}(S, P_f)$



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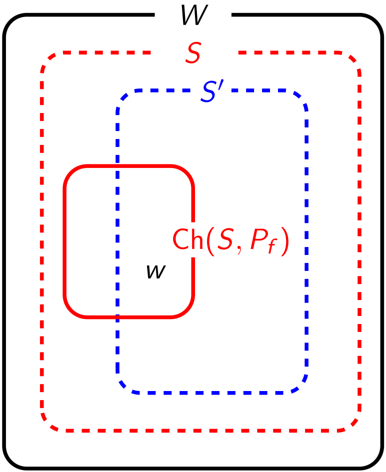

Substitutability **(Kelso and Crawford, 1982)**

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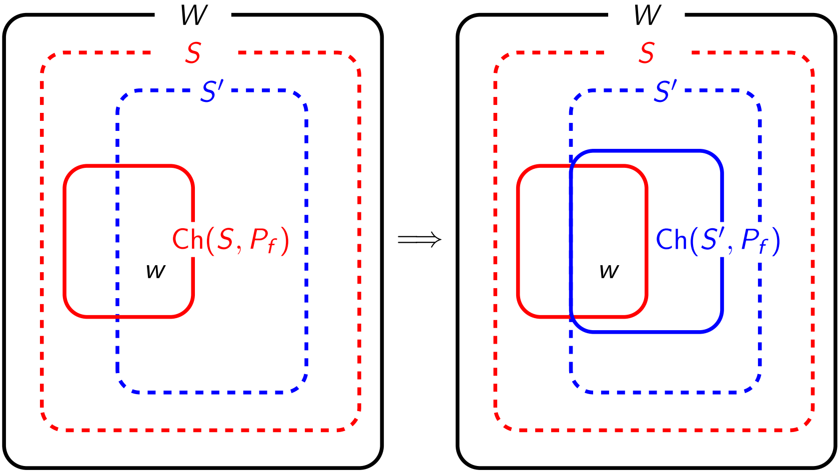
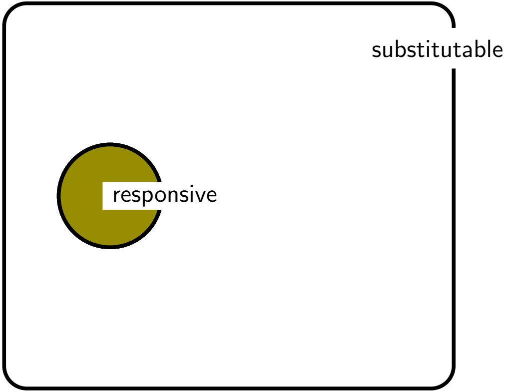


Responsiveness and **Substitutability**



Existence of Stable Matchings

Theorem [Roth, 1984]: P substitutable $\implies \Sigma(P) \neq \emptyset$.

Example: violation of R1 (Martínez et al., 2000)

P_{f_1}	P_{f_2}	P_{f_3}	P_{w_1}	P_{w_2}	P_{w_3}	P_{w_4}
w_3	$w_1 w_2$	w_4	f_1	f_1	f_2	f_2
$w_1 w_3$	$w_1 w_3$	$\emptyset$	f_2	f_2	f_1	f_3
$w_2 w_3$	$w_1 w_4$	$\ldots$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$w_1 w_2$	$w_2 w_3$		$\ldots$	$\ldots$	$\ldots$	$\ldots$
w_1	$w_2 w_4$					
w_2	$w_3 w_4$					
$\emptyset$	w_1					
$\ldots$	w_2					
	w_3					
	w_4					
	$\emptyset$					
	$\ldots$					

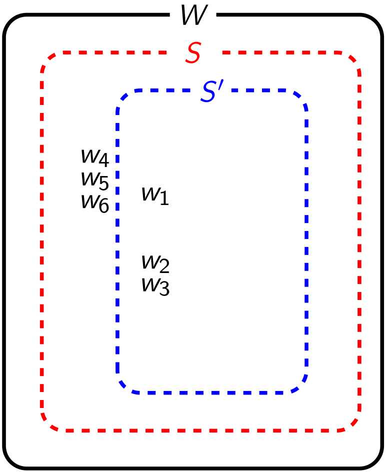
Preferences are substitutable but **R1** does **not** hold:

Clearly, $\mu = (w_3, w_1 w_2, w_4)$ and $\mu' = (w_1 w_2, w_3 w_4, \emptyset)$

are stable matchings, but $|\mu(f_3)| = 1 \neq 0 = |\mu'(f_3)|$.

Cardinal Monotonicity (Alkan, 2002)

$S' \subseteq S \subseteq W$



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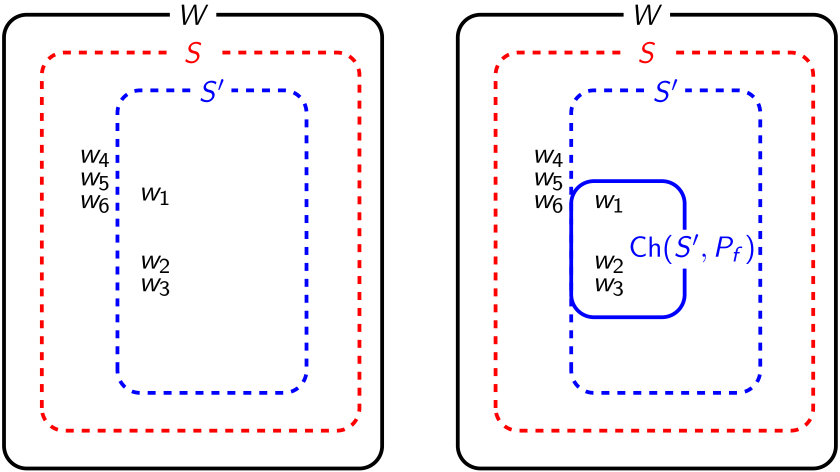
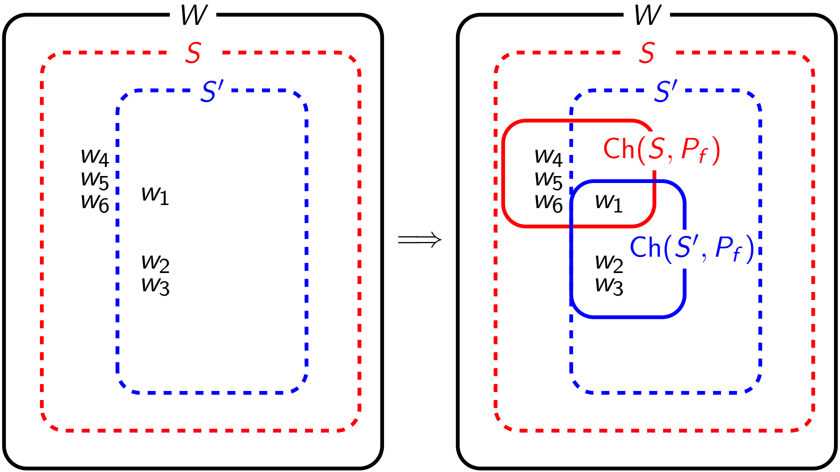


Cardinal Monotonicity (Alkan, 2002)

$$S' \subseteq S \subseteq W \qquad \Longrightarrow \qquad |\,\mathrm{Ch}(S', P_f)\,| \leq |\,\mathrm{Ch}(S, P_f)\,|$$



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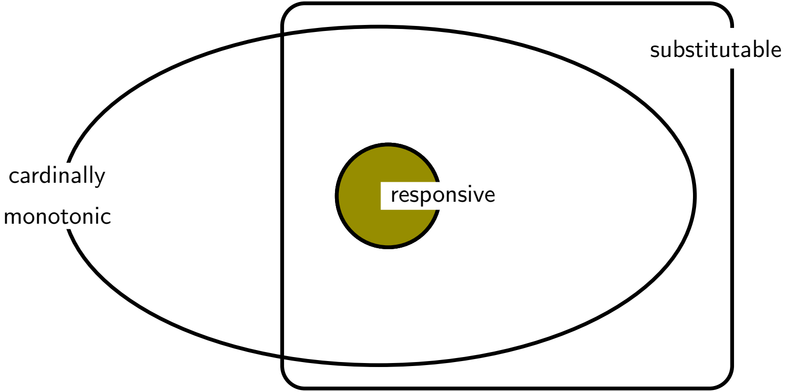


Separability (Martínez et al., 2000)

For each $S \subseteq W$ with $|S| < q_f$
and each $w \in W \backslash S$,

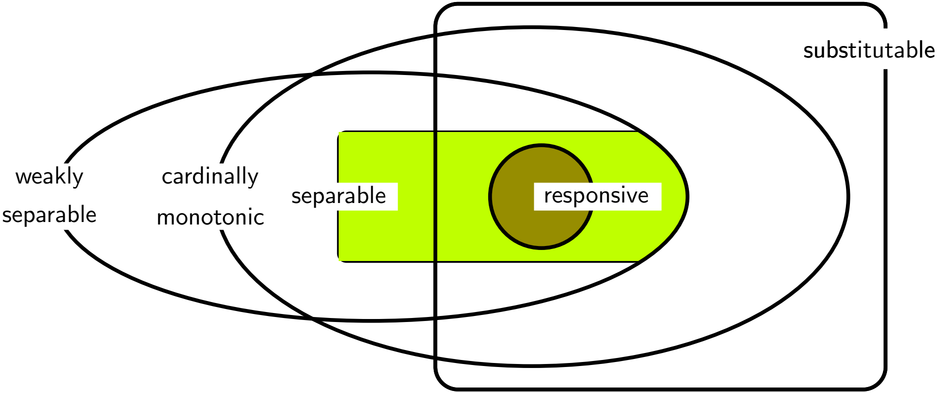
$$S \cup w \quad P_f \quad S \quad \iff \quad w \; P_f \; \emptyset$$

Weak Separability (Klijn and Yazıcı, 2013)

For each $S \subseteq W$ with $|S| < q_f$ and $\mathrm{Ch}(\mathbf{S}, \mathbf{P_f}) = \mathbf{S}$
and each $w \in W \backslash S$,

$$S \cup w \quad P_f \quad S \qquad \Longleftarrow \qquad w \; P_f \; \emptyset$$

Separability and Weak Separability



Quota Filling (Alkan, 2001)

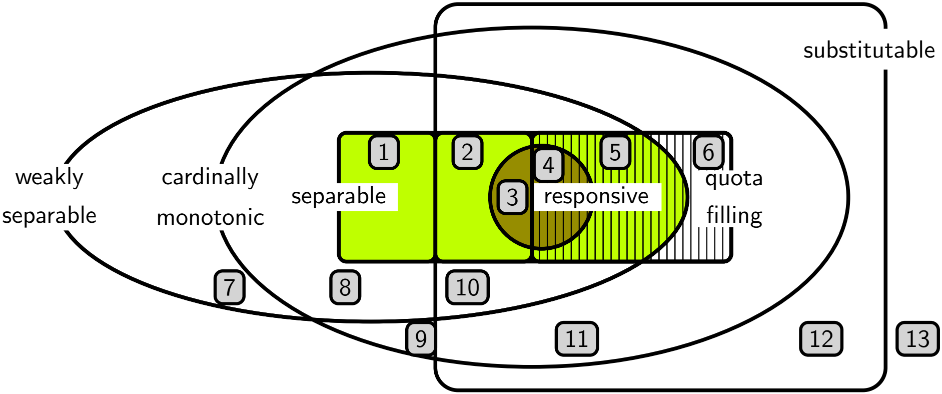
The definition is in two parts:

QF1. Let $\mathbf{k_f} \equiv \max\{\, |\mathrm{Ch}(S, P_f)| \;:\; S \subseteq W \,\}$. For each $S \subseteq W$,

$$|S| \geq \mathbf{k_f} \quad \Longrightarrow \quad |\mathrm{Ch}(S, P_f)| = \mathbf{k_f}.$$

QF2. P_f substitutable.

Inclusion Relations among Preference Domains



Rural Hospital Theorem

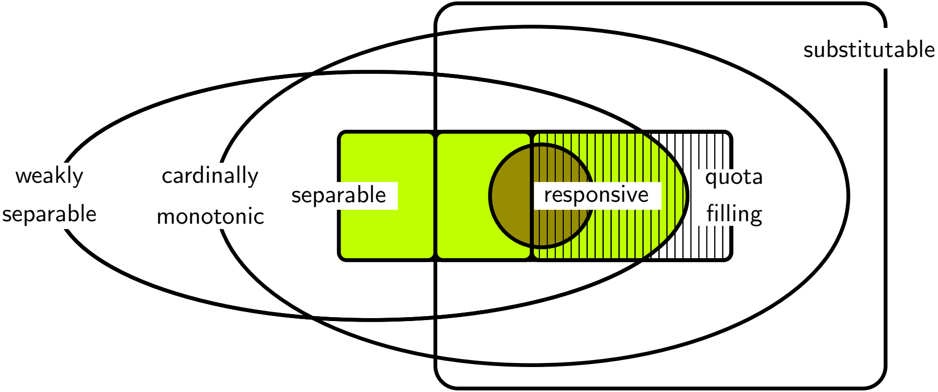
(Recall: $\Sigma(P)$ denotes the set of stable matchings at P.)

- **R1**. For each pair $\mu, \mu' \in \Sigma(P)$ and each $a \in F \cup W$,
$$|\mu(a)| = |\mu'(a)|;$$
- **R2**. For each pair $\mu, \mu' \in \Sigma(P)$ and each $a \in F \cup W$,
$$|\mu(a)| < q_a \implies \mu(a) = \mu'(a).$$

Results, Part I:

Domains for Rural Hospital Theorem

R1


substitutable
weakly
separable
cardinally
monotonic
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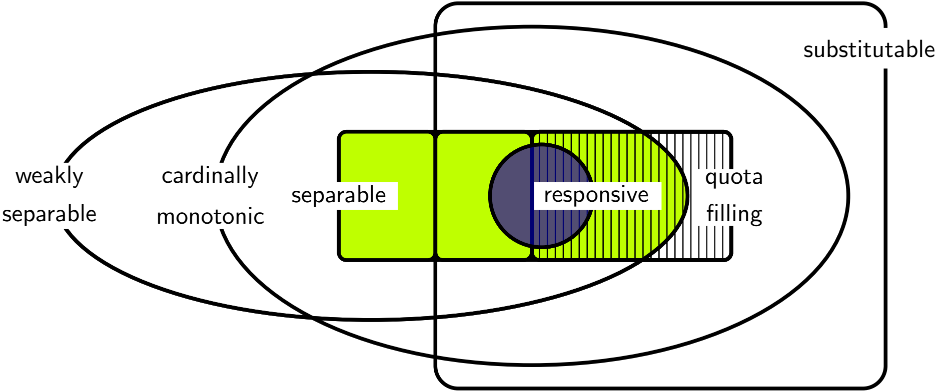
responsive
quota
filling

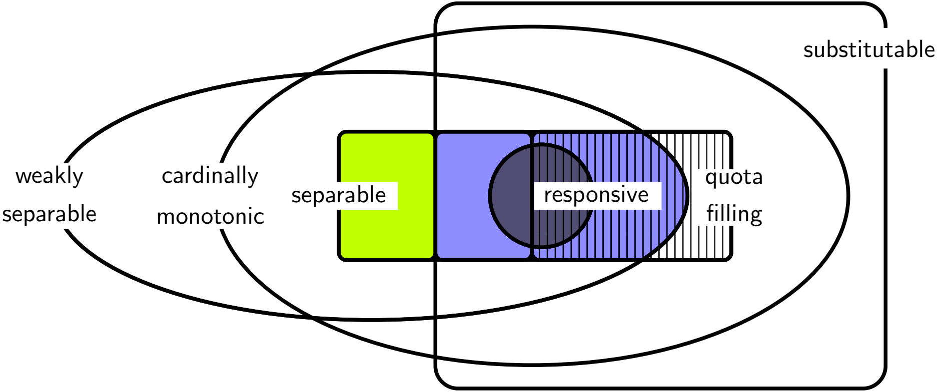
R1

Many–to–**one**, Gale and Sotomayor (1985) and Roth (1984):



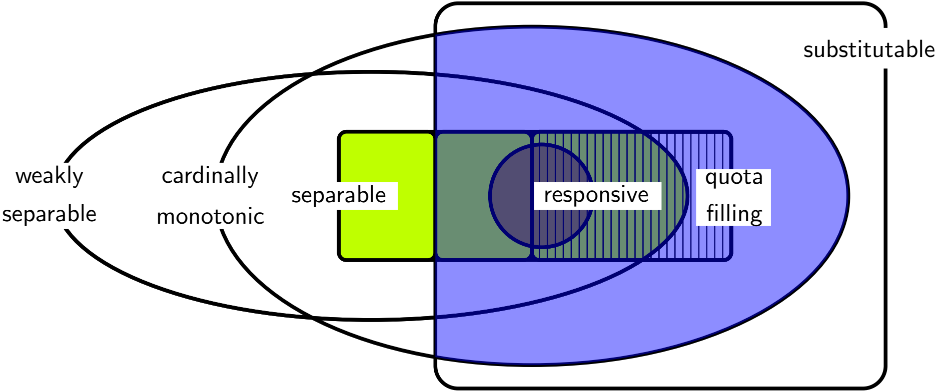
R1

Many–to–**one**, Martínez et al. (2000):

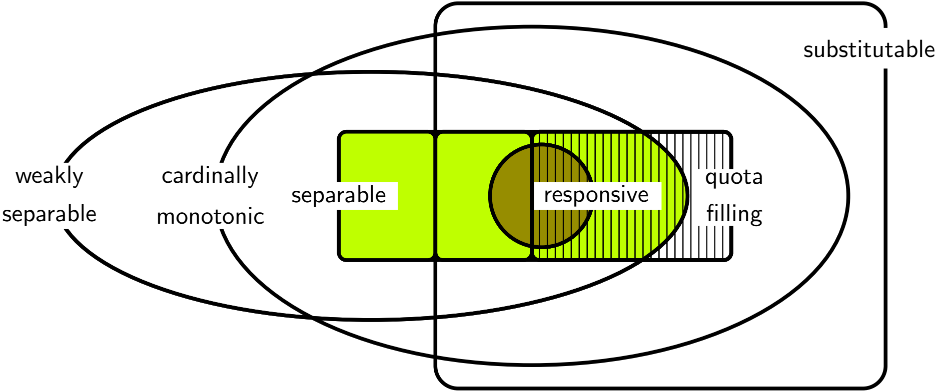


R1

Many–to–**many**, Alkan (2002):

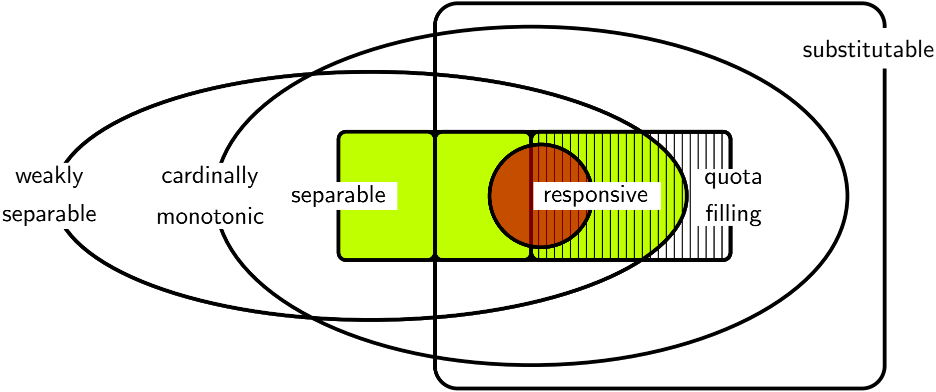


R2


substitutable
weakly separable
cardinally monotonic
separable
responsive
quota filling

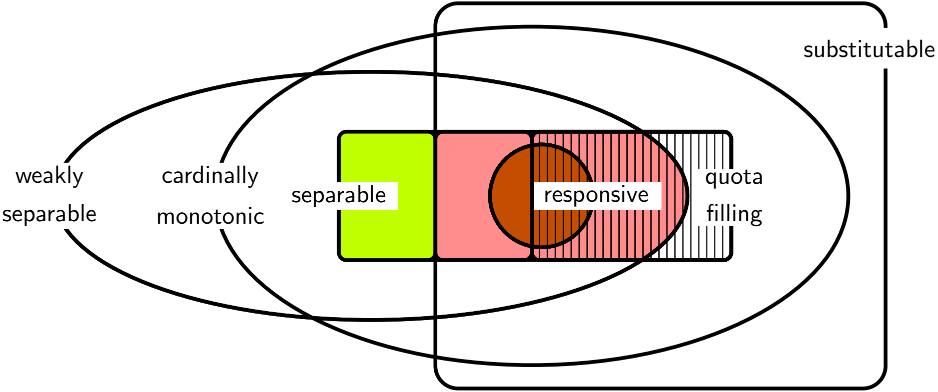
R2

Many–to–**one**, Roth (1986):



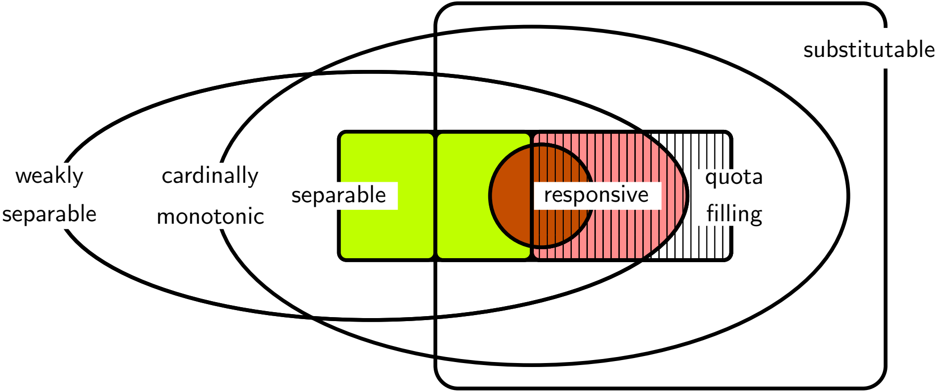
R2

Many–to–**one**, Martínez et al. (2000):

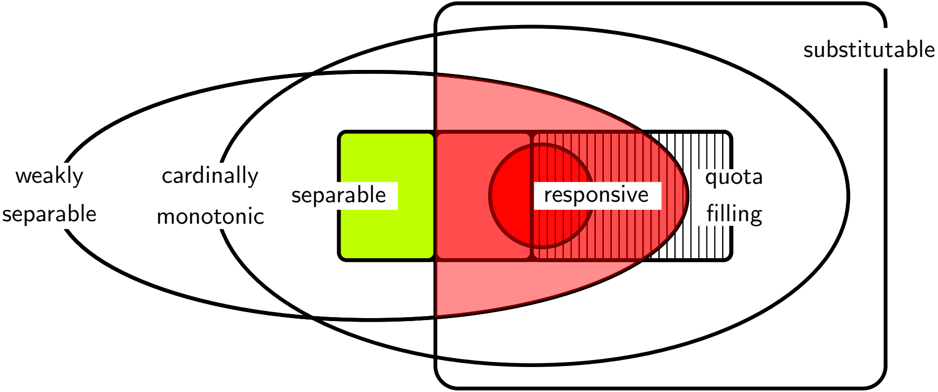


R2

Many–to–**many**, Alkan (1999, 2001):



Many–to–**many**, Klijn and Yazıcı (2013):



Results, Part II:

Maximality of Domains for Rural Hospital Theorem

Definition of Maximality

Let D be a subdomain of the domain of substitutable preferences.

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Definition of Maximality

Let D be a subdomain of the domain of substitutable preferences.

D is **maximal** for property R if

- [for all $a \in F \cup W$, $P_a \in D$] $\Rightarrow$ property R **holds** at P; and
- [$a^* \in F \cup W$ and $P_{a^*} \notin D$] $\Rightarrow$ there is P_{-a^*} such that for all $a \in (F \cup W) \backslash a^*$, $P_a \in D$, and property R does **not hold** at P.

Maximality Results (Klijn and Yazıcı, 2013)

Let preferences be substitutable.

- Cardinal monotonicity is a maximal domain for R1.

Maximality Results (Klijn and Yazıcı, 2013)

Let preferences be substitutable.

- Cardinal monotonicity is a maximal domain for R1.
- Weak separability is a maximal domain for R2.

Maximality of Cardinal Monotonicity for R1

$$\overbrace{\begin{array}{c} \underline{P_{f_1}} \\ \vdots \\ \vdots \\ \text{subs.} + \\ \text{not card. mon.} \end{array}}^{\textit{given} \text{ to us}}$$

Maximality of Cardinal Monotonicity for R1

$$
\overbrace{\begin{array}{c} \underline{P_{f_1}} \\ \vdots \\ \vdots \\ \text{subs.} + \\ \text{not card. mon.} \end{array}}^{\textit{given} \text{ to us}}
\left|
\overbrace{\begin{array}{ccc} \underline{P_{f_2}} & \cdots & \underline{P_{w_m}} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ \text{subs.} + & \text{subs.} + & \text{subs.} + \\ \text{card. mon.} & \text{card. mon.} & \text{card. mon.} \end{array}}^{\text{we can } \textit{construct}}
\right.
$$

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\;\left|\;
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\right.
$$

such that R1 does not hold at P.

Maximality of Weak Separability for R2

$$\overbrace{\begin{array}{c} \underline{P_{f_1}} \\ \vdots \\ \vdots \\ \text{subs.} + \\ \text{not weakly sep.} \end{array}}^{\textit{given}\text{ to us}}$$

Maximality of Weak Separability for R2

given to us	we can *construct*		
$\underline{P_{f_1}}$	$\underline{P_{f_2}}$	$\dots$	$\underline{P_{w_m}}$
$\vdots$	$\vdots$	$\dots$	$\vdots$
$\vdots$	$\vdots$	$\dots$	$\vdots$
subs. $+$ not weakly sep.	subs. $+$ weakly sep.	subs. $+$ weakly sep.	subs. $+$ weakly sep.

Maximality of Weak Separability for R2

given to us	we can *construct*		
$\underline{P_{f_1}}$	$\underline{P_{f_2}}$	$\dots$	$\underline{P_{w_m}}$
$\vdots$	$\vdots$	$\dots$	$\vdots$
$\vdots$	$\vdots$	$\dots$	$\vdots$
subs. $+$ not weakly sep.	subs. $+$ weakly sep.	subs. $+$ weakly sep.	subs. $+$ weakly sep.

such that R2 does not hold at P.

Conclusion

The rural hospital theorem

☹ persists on a larger preference domain than was known so far;

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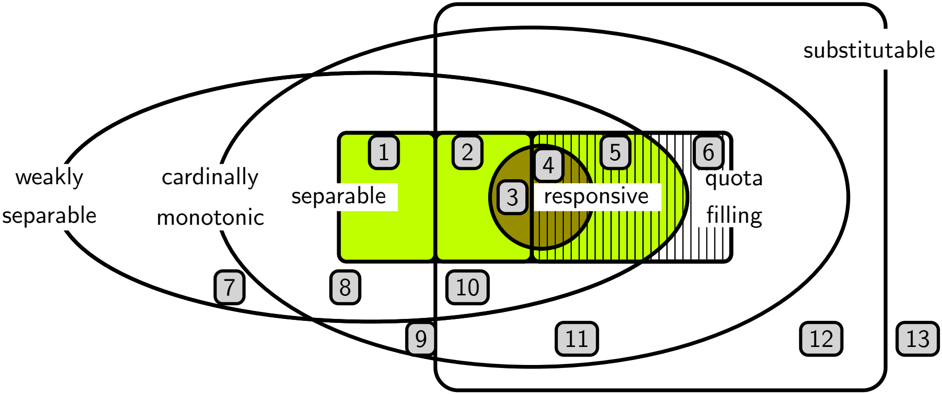
The rural hospital theorem

☹ persists on a larger preference domain than was known so far;

☺ yet, our maximality results suggest its impact is rather limited.

Thank you!

Inclusion Relations among Preference Domains



1. $\{w_1, w_2\}, \{w_3, w_4\}, \{w_1, w_3\}, \{w_1, w_4\}, \{w_2, w_3\}, \{w_2, w_4\}$, $\{w_1\}, \{w_2\}, \{w_3\}, \{w_4\}, \emptyset$.
2. $\{w_1, w_2\}, \{w_1, w_3\}, \{w_1, w_4\}, \{w_2, w_3\}, \{w_3, w_4\}, \{w_2, w_4\}$, $\{w_1\}, \{w_2\}, \{w_3\}, \{w_4\}, \emptyset$ and there is at least one other worker, say w_5.
3. $\{w_1, w_2\}, \{w_1\}, \{w_2\}, \emptyset, \{w_1, w_2, w_3\}, \{w_1, w_3\}, \{w_2, w_3\}, \{w_3\}$ and there are no other workers.
4. $\{w_1, w_2\}, \{w_1\}, \{w_2\}, \emptyset$ and there are no other workers.
5. $\{w_1, w_2\}, \{w_1, w_3\}, \{w_2, w_3\}, \{w_1, w_4\}, \{w_2, w_4\}, \{w_3, w_4\}$, $\{w_2\}, \{w_3\}, \{w_4\}, \{w_1\}, \emptyset$ and there are no other workers.
6. $\{w_1\}, \{w_1, w_2\}, \{w_2\}, \emptyset$ and there are no other workers.
7. $\{w_1, w_4\}, \{w_2, w_3, w_4\}, \{w_4\}, \emptyset$.
8. $\{w_1, w_2, w_3, w_4\}, \{w_1\}, \{w_1, w_2, w_3\}, \emptyset$.
9. $\{w_2, w_3, w_4\}, \{w_1, w_4\}, \{w_2\}, \{w_4\}, \emptyset$.
10. $\{w_1, w_2\}, \{w_1\}, \{w_1, w_3\}, \{w_1, w_2, w_3\}, \{w_2\}, \emptyset$.
11. $\{w_1, w_3\}, \{w_1, w_2, w_3\}, \{w_2, w_3\}, \{w_1\}, \{w_3\}, \{w_2\}, \emptyset, \{w_1, w_2\}$.
12. $\{w_1\}, \{w_2, w_4\}, \{w_2, w_3\}, \{w_4\}, \{w_3\}, \{w_2\}, \emptyset$.
13. $\{w_1, w_4\}, \{w_2, w_3, w_4\}, \{w_2\}, \{w_4\}, \emptyset$.

Example of $\boxed{10}$: substitutable and weakly separable, but not separable

$\mathbf{P_f}$
$w_1 w_2$
w_1
$w_1 w_3$
$w_1 w_2 w_3$
w_2
$\emptyset$
$\dots$

- $q_f = 3$
- P_f substitutable
- P_f weakly separable
- P_f **not** separable since $w_2 \, P_f \, \emptyset$ but $w_1 w_3 \, P_f \, w_1 w_2 w_3$

Example of $\boxed{6}$: quota filling, but not weakly separable

$\mathbf{P_f}$
w_1
$w_1 w_2$
w_2
$\emptyset$

- $k_f = 1$, $q_f = 2$
- P_f quota filling since
  $|\mathrm{Ch}(w_1, P_f)| = |\mathrm{Ch}(w_1 w_2, P_f)| = |\mathrm{Ch}(w_2, P_f)| = 1$
- P_f **not** weakly separable since
  $|\{w_1\}| < q_f$, $\mathrm{Ch}(w_1, P_f) = w_1$, and $w_2 \ P_f \ \emptyset$,
  but $w_1 \ P_f \ w_1 w_2$

Existence of Stable Matchings

Theorem [Roth, 1984]: P substitutable $\Longrightarrow \Sigma(P) \neq \emptyset$.

Proof: Using Gale and Shapley's (1962) Deferred Acceptance algorithm:

1.a. Each firm proposes to its choice set from the set of all workers.

1.b. Each worker accepts *tentatively* its choice set from the set of firms that propose to him/her. Other proposals are rejected.

$$\vdots$$

k.a. Each firm proposes to its choice set from the set of workers that have not rejected it yet.

k.b. Each worker accepts *tentatively* its choice set from the set of firms that propose to him/her. Other proposals are rejected.

The algorithm terminates when no more rejections are issued. The tentative matching becomes final and is stable. $\square$

Example: violation of R2 (Kojima, 2012)

P_{f_1}	P_{f_2}	P_{w_1}	P_{w_2}	P_{w_3}
$w_1 w_3$	$w_1 w_2$	f_2	f_1	f_2
$w_1 w_2$	$w_1 w_3$	$\emptyset$	f_2	f_1
w_1	$w_2 w_3$	$\cdots$	$\emptyset$	$\emptyset$
w_3	w_1		$\cdots$	$\cdots$
w_2	w_2			
$\emptyset$	w_3			
$\cdots$	$\emptyset$			
	$\cdots$			

Preferences are substitutable but **R2** does **not** hold:

$\mu = (w_3, w_1 w_2)$ and $\mu' = (w_2, w_1 w_3)$ are stable matchings

with $|\mu(f_1)| = |\mu'(f_1)| = 1 < 2 = q_{f_1}$ and $\mu(f_1) \neq \mu'(f_1)$.